# Two Approaches to Computing Solutiongs of Stationary Problems in Fluid Dynamics and Magnetohydrodynamics(Mathematical Analysis of Fluid and Plasma Dynamics)

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# Two Approaches to Computing Solutions of Stationary Problems in Fluid Dynamics and Magnetohydrodynamics

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**Introduction.** A wide class of stationary problems in fluid dynamics and magnetohydrodynamics can be formulated in the following way.

Find a function $u(x)$ satisfying the equation

$$\begin{aligned} Lu - g_\sigma(x,u) &= \Lambda(x,u), \quad x \in \Omega \\ u|_{\partial\Omega} &= \phi(x), \end{aligned} \tag{1}$$

where $\Omega$ is a domain in $\mathbf{R}^2$ or $\mathbf{R}^3$. The operator $L$ is a linear positive definite operator in a ceratain Hilbert space specified by a particular problem. The function $g(x,\sigma)$ is usually, although not always, monotone increasing and the subscript $\sigma$ denotes the partial derivative $\partial/\partial\sigma$.

The two classes of problems (and correspondingly, two different approaches to solving these problems) discussed in this paper differ in the way the function $\Lambda(x,\sigma)$, which we call a profile function, is defined. In Section 1 we consider the case when this function is given **explicitly** (up to a small number of parameters). In the case discussed in Section 2 the function $\Lambda(x,\sigma)$ is determined **implicitly** through a solution of an equivalent variational problem with a possibly infinite set of constraints (conservation laws). For both cases we shall present an iterative procedure which generates a sequence converging to a solution globally, i.e. from any feasible initial approximation.

## 1. Explicit Profile Function.

In many stationary problems in fluid dynamics the function $\Lambda(x,\sigma)$ of (1) has the form

$$\Lambda(x,\sigma) = f_\sigma(x,\sigma;\lambda_1,\ldots,\lambda_m), \tag{2}$$

where the function $f(x,\sigma;\lambda_1,\ldots,\lambda_m)$ is explicitly given, the number $m$ is fixed and typically very small and the parameters $\lambda_1,\ldots,\lambda_m$ are unknowns to be found together with the function $u(x)$.

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Below we list several examples of problems of this kind. We have studied these problems extensively with the help of the method which will be described in this section and here we will use them to illustrate the details of its implementation.

**Nonlinear planetary (or Rossby) waves,** see [1]. In [1] the equation (1) appears in which the role of $u$ is played by $\omega$ – the vorticity of the disturbance of the ambient flow, $L \equiv G$ is a Green's operator, $g_\sigma(x,\sigma)$ is a monotone increasing piecewise quadratic function of $\sigma$ for every fixed $x$, $m = 1$ and $f_\sigma(\sigma;\lambda) = \lambda r_\sigma(\sigma)$, where $r_\sigma(\sigma)$ is a linear function of $\sigma$.

**Free-boundary problems in vortex dynamics: vortex pairs, vortex streets, vortex rings with swirl,** see [2, 3, 4]. In these problems the stream function $\psi(x)$ of a flow induced by a system of isolated vorticity regions plays the role of $u$ in (1), $L$ is the Laplacian in various coordinate systems, $g_\sigma(x,\sigma) = 0$ the function $f_\sigma(x,u;\lambda_1,\lambda_2) \equiv f_\sigma(u(x) - \lambda_1\eta(x) - \lambda_2)_+$ where $f(\sigma)$ is a convex function of $\sigma$ and $\eta$ is a given function of $x$ only. The unknown function $u(x)$ is either a scalar function, as in case of vortex pairs and vortex rings with swirl, or a vector function, as in the case of vortex streets. In the latter case (1) becomes a system of nonlinear equations and the definition of the function $f$ is slightly different from the one above but this difference has no affect at all on the presentation of the algorithm and we can ignore it here without the loss of generality.

Other examples can be found in [5] and in [7]. A general step in devising algorithms for solving problems of this type has always been reduction of the corresponding equation (1) with the function $\Lambda(x,u)$ defined in (2) to a variational problem of the form

$$\text{Find} \max_{u \in K \subset H} \Phi(\omega) = E(\omega) - \Psi(\omega), \tag{3}$$

where $E$ is a differentiable convex functional in a Hilbert space $H$.

Specifically, for the examples above these functionals and the set $K$ are as follows.

1) Solitary Rossby Waves.

$$\begin{array}{c} E(\omega) = \int \frac{1}{2}\omega G\omega \, dx_1\, dx_2, \quad \Psi(\omega) = \int g(x,\omega)\, dx_1\, dx_2 \\ K: \ \int r(\omega)\, dx\, dy = R \ \text{ and } \ H = L^2(\Omega) \end{array} \tag{4}$$

2) Vortex Pairs, Rings and Streets. Introducing a function $\omega = -\Delta u$ we replace the equation (1) by the equivalent variational problem (3) with

$$\begin{array}{l} E(\omega) = \int \frac{1}{2}\omega G\omega \, dx_1\, dx_2 \quad \Psi(\omega) = \int f^*(\omega)\, dx_1\, dx_2 \\ K: \ \ \omega \geq 0, \quad \int \omega \, dx_1\, dx_2 = C, \ \ \iint \eta\omega \, dx_1\, dx_2 = P \end{array} \tag{5}$$

where $f^*(\sigma)$ is a convex conjugate function for $f(\sigma)$.

NUMERICAL ALGORITHM

The iterative schemes developed for solving problems above together with many other applications are all based on the following algorithm devised to find a solution of a general problem (3):

$$\omega^0 \in K \quad \text{is an arbitrary function}$$

$$\omega^{k+1} = \max_K \langle E'(\omega^k), \omega \rangle - \Psi(\omega) \tag{6}$$

The principal property of the procedure (6) is its convergence which is based on the following monotonicity property:

$$\Phi(\omega^{k+1}) - \Phi(\omega^k) \geq 0. \tag{7}$$

The proof of (7) follows immediately from (6) and convexity of $E$:

$$\begin{aligned} 0 &\leq \langle E'(\omega^k), \omega^{k+1} - \omega^k \rangle - \Psi(\omega^{k+1}) + \Psi(\omega^k) \\ &\leq E(\omega^{k+1}) - E(\omega^k) - \Psi(\omega^{k+1}) + \Psi(\omega^k) = \Phi(\omega^{k+1}) - \Phi(\omega^k) \end{aligned}$$

Moreover, for a typical case of quadratic $E(u)$

$$\Phi(\omega^{k+1}) - \Phi(\omega^k) \geq E(\omega^{k+1} - \omega^k).$$

The general procedure (6) leads to simple and efficient algorithms for solving concrete problems. For example, in the case of problem (5)

$$\omega^{k+1} = f_\sigma(G\omega^k - \mu_1^{k+1}\eta - \mu_2^{k+1})_+.$$

The implementation of this formula consists of first finding $\psi^k = G\omega^k$ by solving $-\Delta\psi^k = \omega^k$ in $\Omega$ and then computing a pair $\mu_1^{k+1}, \mu_2^{k+1}$ by minimizing a convex function

$$R^k(\mu_1, \mu_2) = \mu_1 + \mu_2 + \int f(\psi^k - \mu_1\eta - \mu_2)_+ dx_1 dx_2, \quad \mu_1, \mu_2 \geq 0.$$

Clearly, each step of this procedure can be easily implemented.

## 2. Multiconstrained Variational Problems.

Let us now consider the following variational problem:

$$\begin{aligned} \text{Find} \quad & \min E(u) = \textstyle\int_\Omega \frac{1}{2} uLu\, dx \quad \text{subject to constraints} \\ & \textstyle\int_\Omega (u - \sigma)_+ dx = \beta(\sigma), \quad \sigma \in [0, \infty]. \end{aligned} \tag{8}$$

The connection between problem (1) and (8) follows from the following simple observation. Let $u^*$ be a solution of (8) and let $\lambda(\sigma)$ be the corresponding Lagrange multipliers. Then, formally,

$$Lu^*(x) = \int_0^\infty \lambda(\sigma)\chi_{\{u^*(x)>\sigma\}}d\sigma = \int_0^{u^*(x)} \lambda(\sigma)\, d\sigma.$$

Introducing $\Lambda(\sigma)$ such that $\Lambda'(\sigma) = \lambda(\sigma)$ we obtain that

$$Lu = \Lambda(u^*) \tag{9}$$

which is the equation of the type (1). The difference is that now the nonlinearity $\Lambda(u)$ is not prescribed *a priori*, but rather it follows from the conservation laws (constraints). We note that the argument utilized above to obtain (9) is only formal and was used solely for the illustration purposes, since the validity of the Lagrange multipliers rule in this case of infinitely many constraints is still an open question. In order to make our approach rigorous, we "discretize" the infinite constraint set. For this purpose we introduce a partition $0 \le \sigma_0 < \sigma_1 < \ldots < \sigma_{n-1} < \sigma_n = \infty$. Then we integrate the constraints over each subinterval of this partition to obtain the multiconstrained minimization problem

$$\begin{cases} \int_\Omega \frac{1}{2} uLu\, dx \to \min \quad \text{over} \\ F_i(u) = \int_\Omega f_i(u)\, dx = \gamma_i \quad (i = 1, \ldots, n) \end{cases} \tag{10}$$

where

$$f_i(s) := \frac{1}{2}(s - \sigma_{i-1})_+^2 - \frac{1}{2}(s - \sigma_i)_+^2 = \int_{\sigma_{i-1}}^{\sigma_i} (s - \sigma)_+ d\sigma.$$

and

$$\gamma_i := \int_{\sigma_{i-1}}^{\sigma_i} \beta(\sigma)\, d\sigma.$$

The corresponding variational equations for a solution of (10) are

$$Lu = \Lambda_n(u) \quad \text{with} \quad \Lambda_n(u) = \sum_{i=1}^{n} \lambda_i f_i'(u), \tag{11}$$

where $\lambda_i$ are the Lagrange multipliers. From the definition of the functions $f_i$ it follows that the function $\Lambda_n(\sigma)$ is piecewise linear on $[0, \infty]$ and on every subinterval $[\sigma_{i-1}, \sigma_i]$ its slope is $\lambda_i$. Thus, $\Lambda_n$ can be interpreted as a piecewise linear approximation of the function $\Lambda(\sigma)$ in (9). The significance of the variational approach presented above is evident, since it allows one to describe a physical phenomenon in terms of the conserved quantities; this formulation is sometimes more natural and less arbitrary. In addition to being conceptually interestig, this approach is useful for solving a number of important problems. One of them is presented below.

**Equilibrium magnetohydrodynamics.** Let $\Omega \subseteq \mathbb{R}^2$ be the cross-section of a cylindrical domain $\Omega \times \mathbb{R}$, and let $(x_1, x_2)$ denote the variable point in $\Omega$, and let $x_3$ be the ignorable coordinate. Consider the minimization problem

$$\begin{cases} \int_\Omega [\frac{1}{2}|\nabla u|^2 + h_1(v_1) + h_2(v_2)]\, dx \to \min \text{ over} \\ \int_\Omega v_1(u-\sigma)_+ \, dx = \beta_1(\sigma) \\ \int_\Omega v_2(u-\sigma)_+ \, dx = \beta_2(\sigma), \quad (\sigma_0 \le \sigma < +\infty) \end{cases} \tag{12}$$

where the admissible triple $(u, v_1, v_2)$ belongs to $H_0^1(\Omega) \times L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ for some $1 < r_1, r_2 < +\infty$. The given functions $h_1$ and $h_2$ are assumed to be smooth and strictly convex with $h_\ell(0) = h_\ell'(0) = 0$ and $h_\ell(z) = O(|z|^{r_\ell})$ as $|z| \to \infty$ $(\ell = 1, 2)$. The two infinite family of constraints are parametrized by $\sigma \in [\sigma_0, +\infty)$, and $\beta_1(\sigma)$ and $\beta_2(\sigma)$ are given data.

The physical interpretation of (1) is as follows. The magnetic field $\mathbf{B} = (B^1, B^2, B^3)$, which is independent of $x_3$, satisfies $\nabla \cdot \mathbf{B} = 0$ in $\Omega \times \mathbb{R}$ and hence admits a representation $\mathbf{B} = (u_{x_2}, -u_{x_1}, v_1)$, where $u$ is the flux function (or stream function) for its poloidal part and $v_1$ is its toroidal part. The magnetic energy density (per unit volume) is then $\frac{1}{2}|\mathbf{B}|^2 = \frac{1}{2}|\nabla u|^2 + \frac{1}{2}v_1^2$. The mass density $\rho$ of the plasma is represented by $v_2$. The internal energy density (per unit volume) is given by $\rho^\gamma/(\gamma - 1)$, in accordance with the polytropic law $p = \rho^\gamma$ with $p$ denoting pressure. Therefore, the objective functional represents total (potential) energy when we put

$$h_1(v_1) = \frac{1}{2}v_1^2, \quad h_2(v_2) = \frac{v_2^\gamma}{\gamma - 1}.$$

The interpretation of the constraints relies on differentiating them with respect to the parameter $\sigma$, for then there results

$$\int_{\{u>\sigma\}} v_1 \, dx = -\beta_1'(\sigma), \quad \int_{\{u>\sigma\}} v_2 \, dx = -\beta_2'(\sigma).$$

All of these integrals are extended over the interior of a (cylindrical) flux surface $\{u = \sigma\}$ - that is, a flux tube $\{u > \sigma\}$. The evolution equations of ideal MHD require that each flux tube must move with the flow preserving its flux and mass. It is readily verified that the above integrals are, respectively, the toroidal flux and mass of the flux tube $\{u > \sigma\}$, and hence they are conserved quantities. The conservation of poloidal fluxes is implicit in the parametrization which uses the values of the flux function $u$.

In our future work we will deal with the full problem (12). The model problem (8) (or its discretized version (10)) discussed here corresponds to

the case of incompressible plasma ( $v_2 = 1$ – uniform density – in (12) ) with purely poloidal magnetic field ($v_1 = 0$). The operator $L \equiv -\Delta$.

In [6] we have introduced and justified a computational algorithm for solving problems of type (10).

NUMERICAL ALGORITHM

Let $u^0$, be an arbitrary initial function such that $F_i(u^0) \geq \gamma_i$. Then, for the iteration number $k = 0, 1, \ldots$ we define

$$\begin{cases} u^{k+1} = \arg\min E(u) + \tau\|u\|_2^2 \text{ over} \\ F_i(u^k) + \langle F_i'(u^k), u - u^k \rangle \geq \gamma_i, \end{cases} \tag{13}$$

where $F_i'$ denote Fréchet derivatives in $u$ and $\tau$ is a fixed positive parameter. This procedure can be efficiently implemented.

Indeed each iteration of the procedure (13) can be carried out in two steps.

Step 1: Find $z_i^{k+1}$ as a solution of a simple linear (elliptic) problem

$$E'(z_i^{k+1}) = F_i'(u^k), \quad i = 1, \ldots, n;$$

Step 2: Compute the next iterate $u^{k+1} = \sum_{j=1}^n \mu_j^{k+1} z_j^{k+1}$, where the vector $\mu^{k+1} = (\mu_1^{k+1}, \ldots, \mu_n^{k+1})$ is a solution of the $n$-dimensional convex quadratic optimization problem:

$$\mu^{k+1} = \arg\min \sum \frac{1}{2} a_{ij}^k \mu_i \mu_j - \sum c_j^k \mu_j, \quad \text{over } \mu_i \geq 0, \quad \text{where}$$

$$a_{ij}^k = \langle F_i'(u^k), z_j^{k+1} \rangle, \quad c_i^k = \gamma_i - F_i(u^k) + \langle F_i'(u^k), u^k \rangle.$$

The first step essentially consisits of solving $n$ classical Poisson equations (for a common elliptic operator), which can be done efficiently using standard software. Since the number $n$ of partition levels $\sigma_i$ is an order of magnitude smaller than the number of discretization nodes in the domain $\Omega$, the second step presents a simple low-dimensional optimization problem which can also be efficiently solved by standard methods.

In [6] the proof of convergence of the procedure (13) and the estimates on $\tau$ are given.

In the conclusion we note that both approaches discussed in this paper provide us with considerable flexibility in investigating eigenvalue type problems (1). We are no longer restricted by the complexity of the problem (8) and can freely choose either the representation (1,2) or (8,9) justifying our choice only by relevant physical considerations.

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